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MINIMUM EVAPORATION IN TWO-PHASE FLOWS

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Abstract—Design variables for attaining minimal evaporation rate by properly varying the cross-sections of ducts transporting gas-droplets flows are investigated for low concentrations of droplets. Applying variational techniques, the cases in which such a minimum exists are established, together with the general behavior of the mathematical solution. The exact longitudinal distribution of velocity (and cross-section area) is obtained numerically by the gradient method.

NOMENCLATURE

C_D, drag coefficient;
C_p, $= c_p/\gamma R_g$, non-dimensional specific heat of liquid;
C_{pg}, $= c_{pg}/\gamma R_g$, non-dimensional specific heat at constant pressure of the gas mixture;
C_{pv}, $= c_{pv}/\gamma R_g$, non-dimensional specific heat at constant pressure of the vapors;
D, $= d/\rho_t$, non-dimensional density of liquid;
F_1, F_2, F_3, functions, defined by equations (2), (3), (4);
H, functional defined by equation (8);
H_{fg}, $= h_{fg}/U_t^2$, non-dimensional heat of evaporation;
L, $= l/r_0$, non-dimensional length of the duct;
Nu, Nusselt number for heat transfer;
Nu_D, Nusselt number for mass transfer;
P, $= p/\rho_t U_t^2$, non-dimensional pressure;
Pr, Prandtl number;
P_{v0}, $= p_{v0}/\rho_t U_t^2$, non-dimensional vapor pressure;
r_0, initial radius of droplets (at the duct entrance);
R, $= r/r_0$, non-dimensional radius of droplet;
Re, Reynolds number based on droplet diameter;
R_g, gas constant;
Sc, Schmidt number;
T_t, gas temperature at the throat;
U, $= u/U_t$, non-dimensional gas velocity;
U_t, gas velocity at the throat;
V, $= v/U_t$, non-dimensional droplet velocity;
W_p, $= w_p/\rho_t U_t$, non-dimensional mass flow rate of droplets;
X, $= x/r_0$, non-dimensional distance along the duct (measured from the entrance).

Greek symbols

γ, specific heat ratio of the gas mixture;
θ_g, $= T_g/T_t$, non-dimensional gas temperature;
θ_p, $= T_p/T_t$, non-dimensional droplet temperature;
λ_i, Lagrange multipliers;
μ, viscosity coefficient;
μ^*, non-dimensional viscosity coefficient defined by equation (5);
ρ, $= \rho/\rho_t$, non-dimensional density of the gas mixture;
ρ_t, gas density at the throat;
χ_{vg}, vapor mole-fraction in the gas mixture;
χ_{vs}, vapor mole-fraction at droplet surface.

INTRODUCTION

VERY few studies have, so far, dealt with the optimization of two-phase flows; Marble [1] calculated optimal contours of rocket nozzles for one-dimensional flows where maximal specific impulse for a given nozzle length is sought. Hoffman and Thomson [2] and Kraiko and Osipov [3] estimated maximal thrust obtained with an axi-symmetric nozzle for given length and area. However, these works are restricted to solid particles, without mass transfer between the phases. In practice, many two-phase flows take place inside transport devices with differing cross-sections and involve evaporation of droplets or aerosols, which affects the computations. Estimation of the total evaporation of the dispersed phase at a certain cross-section in a transport device depends on the variation of the flow properties with distance up to that point. These variations must be taken into account whenever it becomes desirable to minimize evaporation losses along the duct. In the present work we develop a new method to estimate the conditions which lead to such minimal evaporation losses. This involves distribution of velocity and temperature profiles to be obtained by properly varying the cross-section areas of the duct.

The total evaporation of the entrained droplets is given by integration of local evaporation rates along the duct. The mathematical problem at hand is, therefore, to minimize this integral. Application of variational techniques to minimize this integral would thus lead to a differential equation for the velocity distribution (which, however, cannot be solved analytically for most practical cases). In the present work we calculate the exact solution numerically, employing the gradient method, with a high speed computer.

FORMULATION OF THE PROBLEM

The extremum problem to be solved here is for a one-dimensional gas flow entraining a low concentration of spherical droplets of uniform initial radii. We assume that there is no mass or energy transfer from the bulk of the two-phase flow to the surroundings, and as the droplets concentration is very low, the flow is isentropic. Thus gas properties at each point depend only on initial conditions and local cross-section area, and can be calculated by the usual one-dimensional isentropic flow relations. All properties are made non-dimensional (dividing them by the gas critical conditions at a fictitious throat, which corresponds to the given stagnation conditions). Since local cross-sectional area becomes a single-valued function of the flow velocity, other gas properties are to be expressed in terms of U, viz.:

$$\theta_g = \frac{\gamma+1}{2} - \frac{\gamma-1}{2}U^2 \tag{1a}$$

$$\rho = \theta_g^{1/(\gamma-1)} \tag{1b}$$

$$P = \frac{1}{\gamma}\theta_g^{\gamma/(\gamma-1)}. \tag{1c}$$

The particle properties vary along the duct due to mass, energy and momentum exchanges with the gas phase. The ordinary differential equations which describe these variations are, for diffusion controlled evaporation:

$$\frac{dW_p}{dX} = F_1(W_p, \theta_p, V, U) = -\frac{3W_p\mu^* Nu_D}{2DR^2VSc}\frac{\chi_{vs}-\chi_{vg}}{1-\chi_{vs}} \tag{2}$$

$$\frac{d\theta_p}{dX} = F_2(W_p, \theta_p, V, U) = \frac{3C_{pg}\mu^* Nu}{2DR^2VC_pPr}(\theta_g - \theta_p) + \frac{H_{fg}}{W_pC_p}\frac{dW_p}{dX} \tag{3}$$

$$\frac{dV}{dX} = F_3(W_p, V, U) = \frac{3}{8}C_D\frac{\rho(U-V)|U-V|}{DRV} \tag{4}$$

where $R = (W_p/W_{po})^{1/3}$,

$$\mu^* = \frac{\mu}{r_0U_t\rho_t} \tag{5}$$

and χ_{vs} is assumed to depend on the droplet temperature according to the Clausius–Clapeyron equation

$$\chi_{vs} = \frac{P_{vo}}{P}\exp\left[\gamma H_{fg}\left(1 - \frac{1}{\theta_p}\right)\right]. \tag{6}$$

The vapor mole fraction in the gas mixture χ_{vg}, is assumed to be constant for low concentration of droplets.

The total mass transfer at distance L is given by

$$W_p - W_{po} = \int_0^L \left(\frac{dW_p}{dX}\right)dX. \tag{7}$$

This integral is to be minimized, with equations (2)–(4) as differential constraints, which form together a Lagrange type problem. A functional H is now defined by

$$H = \frac{dW_p}{dX} + \lambda_1\left(F_1 - \frac{dW_p}{dX}\right) + \lambda_2\left(F_2 - \frac{d\theta_p}{dX}\right) + \lambda_3\left(F_3 - \frac{dV}{dX}\right) \tag{8}$$

where $\lambda_i(X)$ are Lagrange multipliers. H is, therefore, a function of X, the four dependent variables and their first derivatives. The Euler equations for this case are

$$\frac{\partial H}{\partial Y_j} - \frac{d}{dX}\left(\frac{\partial H}{\partial Y_j'}\right) = 0, \quad (j = 1,2,3,4). \tag{9}$$

Applying these equations to the functional H results in the following:

$$\lambda_1\frac{\partial F_1}{\partial W_p} + \lambda_2\frac{\partial F_2}{\partial W_p} + \lambda_3\frac{\partial F_3}{\partial W_p} + \frac{d}{dX}\lambda_1 = 0 \tag{10a}$$

$$\lambda_1\frac{\partial F_1}{\partial \theta_p} + \lambda_2\frac{\partial F_2}{\partial \theta_p} + \frac{d}{dX}\lambda_2 = 0 \tag{10b}$$

$$\lambda_1\frac{\partial F_1}{\partial V} + \lambda_2\frac{\partial F_2}{\partial V} + \lambda_3\frac{\partial F_3}{\partial V} + \frac{d}{dX}\lambda_3 = 0 \tag{10c}$$

$$\lambda_1\frac{\partial F_1}{\partial U} + \lambda_2\frac{\partial F_2}{\partial U} + \lambda_3\frac{\partial F_3}{\partial U} = 0. \tag{10d}$$

This set can be considerably simplified by expressing λ_1 as a function of λ_2 [from (10b)] and then λ_3 can also be expressed in terms of λ_2 only [from (10d)]. Introducing these results into (10 a,c) yields two second-order differential equations for λ_2. The condition that these two second-order differential equations for λ_2 are consistent gives rise to a second-order ordinary differential equation, which with equations (2)–(4) makes a set of four simultaneous equations for the variables W_p, θ_p, V, U.

BOUNDARY CONDITIONS

This set requires three boundary conditions for equations (2)–(4) and two for the additional equation. The initial conditions of the particles are their known entry conditions, i.e. their velocity, temperature and radii. Possible combinations of fixed or variable boundary conditions for the gas velocity U are now to be investigated.

When the initial and final velocities are specified, the set supplemented with the required boundary conditions can be solved for optimal velocity distributions between the two end values. In other cases the initial velocity is known, and the end value can vary. From the family of solutions, starting from the given value, the end value causing minimal evaporation must be chosen. In the most general case, the initial conditions are also variable and the optimal gas entry velocity must also be determined.

For variable end values we require that

$$\frac{\partial H}{\partial Y_i'} = 0 \quad \text{at} \quad X = 0, L. \tag{11}$$

Applying (13) to the functional (8) one obtains

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 = \lambda_3 &= 0 \quad \text{at} \quad X = 0, L \end{aligned} \tag{12}$$

which, on substitution into (10d), gives

$$\frac{\partial F_1}{\partial U} = 0 \quad \text{at} \quad X = 0, L. \tag{13}$$

By differentiating (2) with respect to U, for Nusselt number expressed as $Nu_D = 2 + 0{\cdot}6Sc^{1/3}Re^{1/2}$ (with a constant Schmidt number), one obtains

$$\frac{\partial F_1}{\partial U} = -\frac{3\mu^* W_p}{2DR^2VSc(1-\chi_{vs})} \times \left\{ 0{\cdot}3Sc^{1/3}\frac{\sqrt{Re}}{U-V}\left[1 - \frac{U}{\theta_g}(U-V)\right](\chi_{vs}-\chi_{vg}) + \gamma Nu_D \frac{U}{\theta_g}\frac{(1-\chi_{vg})\chi_{vs}}{1-\chi_{vs}} \right\} = 0. \tag{14}$$

Although this equation relates the gas velocity U to the particle parameters V, θ_p, R at the duct ends, it does not do so explicitly at $X = L$, as the parameters depend on the velocity distribution, which is as yet unknown. At $X = 0$, on the other hand, the initial values are known, and the optimal initial gas velocity can be calculated from (14). The general plot of F_1 vs U is shown in Fig. 1, and we look for the possibility of getting $\partial F_1/\partial U = 0$ at the ends. At $U > V$ the derivative is always negative, and at $U = V$, F_1 reaches an absolute minimum with a discontinuous derivative (since Reynolds number if proportional to $|U - V|$). Equation (14) can be satisfied only for $U < V$, and not for any combination of initial conditions, e.g. for a given set of initial conditions the entrance velocity of the gas might have an optimal value much lower than the entry velocity of the droplets. In any case, the evaporation calculated for this optimal velocity distribution must be compared with the results for the absolute minimum $U = V$, which is always valid.

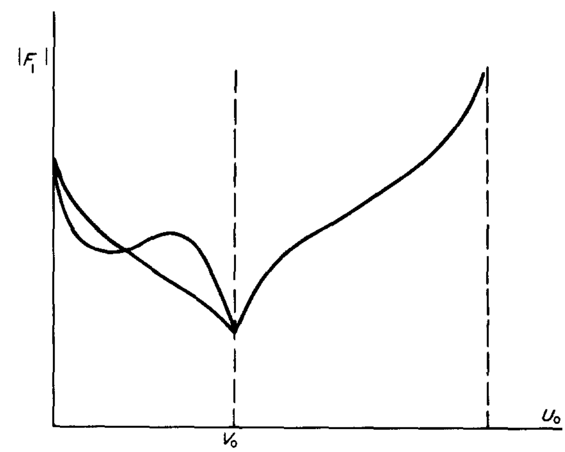

FIG. 1. Evaporation rate vs initial gas velocity.

For a given initial gas velocity, an optimal velocity distribution $U = U(X)$ for minimum evaporation can be obtained with free final velocity expressed by equation (14), but not directly. A first rough estimate of the final velocity is obtained by assuming $Nu_D = 2$ (i.e. zero relative velocity between the phases). In this case the derivative is

$$\frac{\partial F_1}{\partial U} = \frac{3(\gamma-1)W_p\mu^*}{DR^2VSc} U \frac{\partial}{\partial \theta_g}\left(\frac{\chi_{vs}-\chi_{vg}}{1-\chi_{vs}}\right) = 0. \tag{15}$$

As the derivative with respect to θ_g cannot vanish, the extremum condition becomes

$$U = 0 \quad \text{at} \quad X = L. \tag{16}$$

That is, to arrive at a minimum evaporation for given initial conditions, the flow must be decelerated along the duct to reach stagnation at the end.

RESULTS AND CONCLUSIONS

Instead of solving the simultaneous set of four differential equations, we used a computer program, designed to solve optimization problems of this type by the gradient method. Figure 2 presents the numerical results for droplets initial conditions $V = 0{\cdot}2$, $\theta_p = 1{\cdot}0$, $R = 1{\cdot}0$ moving along a duct of length $L = 1000$ where the initial gas velocity is $U_0 = 0{\cdot}5$. Other constants are: $\gamma = 1{\cdot}4$, $C_{pg} = 2{\cdot}5$, $C_p = 5{\cdot}0$, $D = 560$, $\mu^* = 0{\cdot}0225$, $H_{fg} = 5{\cdot}0$ and $Sc = Pr = 1$. The drag coefficient has been expressed as $C_D = 0{\cdot}48 + 28Re^{-0{\cdot}85}$. The figure

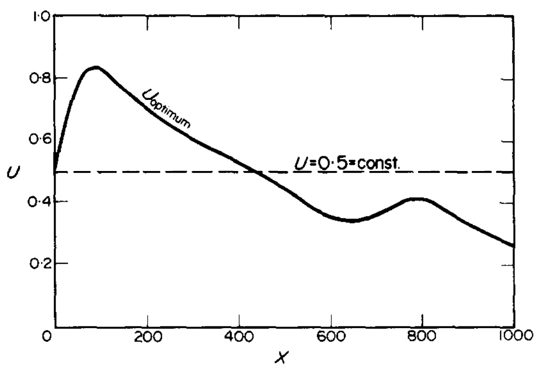

FIG. 2. Velocity distribution for minimum evaporation ($V = 0{,}2$, $U_0 = 0{\cdot}5$).

shows that the velocity must decrease towards the duct end, as was also deduced by assuming constant Nu_D. However, in the exact case a finite velocity is predicted instead of zero. The oscillation obtained between $X = 600$ to 800 starts when U and V approach equality. This could result from the discontinuities of the derivatives, as previously explained, and is insignificant physically.

The percentage of evaporation obtained with the optimal velocity distribution is shown in Fig. 3, where a comparison is also made with the evaporation for constant velocity $U = 0{\cdot}5$. In the latter case the total evaporation is 9·95 per cent while it decreases to 8·65 per cent with the optimal velocity distribution, i.e. a relative reduction of 13 per cent.

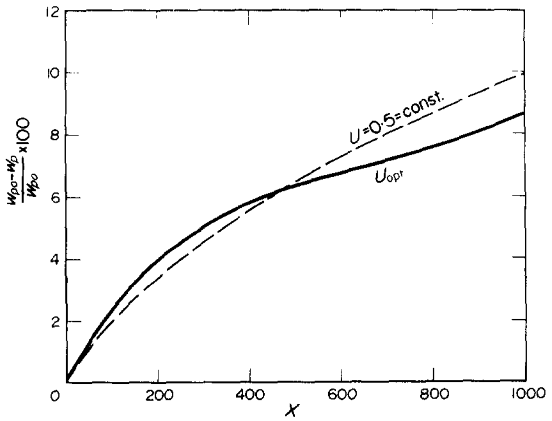

FIG. 3. Evaporation rates for optimum velocity distribution and for constant gas velocity.

Two arbitrary velocity distributions, bounding the optimal one, were also chosen in order to check the validity of the numerical optimization. Evaporation was then found to be higher than in the optimum case.

The present analysis leads to the following conclusions:

(1) For a variable entry velocity of the gas, a constant area duct with $U = V_0$ gives a minimum in total evaporation.

(2) For a fixed entry velocity an optimal contour can be designed according to equations (2)–(4) and (10).

For higher mass-flow rates of particles, the gas is affected by the entrained droplets, so that the flow is no longer isentropic and the full conservation equations must be taken into account. The treatment of this case is different and will be reported elsewhere [4].

REFERENCES

1. F. E. Marble, Nozzle contours for minimum particle-lag loss, *AIAA Jl* **1**, 2793–2801 (1963).
2. J. D. Hoffman and H. D. Thomson, Optimum thrust nozzle contours for gas particle flows, *AIAA Jl* **5**, 1886–1887 (1967).
3. A. N. Kraiko and A. A. Osipov, On the solution of variational problems of supersonic flows of gas with foreign particles, *Prikl. Mat. Mekh.* **32**, 596–605 (1968). English translation (P.M.M.) *J. Appl. Math. Mech.* **32**, 617–627 (1968).
4. Y. Narkis and B. Gal-Or, Two-phase flow through normal shock wave. To be published.

EVAPORATION MINIMALE DANS DES ECOULEMENTS DIPHASIQUES

Résumé—On étudie différents dispositifs pour atteindre le taux minimal d'évaporation en faisant varier convenablement les sections droites de tubes traversés par des écoulements gazeux chargés de gouttes à faible concentration. En appliquant des techniques variationnelles on établit les cas dans lesquels un tel minimum existe et on associe le comportement général de la solution mathématique. On obtient numériquement, par la méthode du gradient, la distribution de vitesse (et l'aire de la section droite).

VERDAMPFUNGS-MINIMUM IN ZWEIPHASENSTRÖMUNGEN

Zusammenfassung—Um den Einfluß von Auslegungsparametern im Hinblick auf minimale Verdampfung in Gas-Tropfen-Strömungen zu ermitteln, wird der Strömungsquerschnitt in geeigneter Weise variiert. Die Untersuchung wird für geringe Tropfenkonzentrationen durchgeführt. Unter Anwendung der Variationstechnik werden die Fälle, für die ein Minimum existiert, festgelegt in Übereinstimmung mit der allgemeinen mathematischen Lösung. Die Geschwindigkeitsverteilung in Längsrichtung wird numerisch mit Hilfe der Gradientenmethode gewonnen.

ОПРЕДЕЛЕНИЕ МИНИМАЛЬНОЙ СКОРОСТИ ИСПАРЕНИЯ В ДВУХФАЗНЫХ ПОТОКАХ

Аннотация — Изучены расчетные переменные, используемые для определения минимальной скорости испарения путем соответствующего изменения поперечных сечений труб, переносящих двухфазные потоки газ–жидкость при малых концентрациях капель. С помощью вариационных методов определяются случаи, в которых существует такая минимальная скорость испарения, а также устанавливается общее поведение математического решения. С помощью градиентного метода численно получено точное продольное распределение скорости (и площади поперечного сечения).